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Positive solutions of the p-Laplace equation with singular nonlinearity

Ahmed Mohammed

Department of Mathematical Sciences, Ball State University, Muncie, IN 47306, USA

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Abstract

For a given bounded domain Ω in $\mathbb{R}^n$ with $C^{1,\varpi}$ boundary for some $0 < \varpi < 1$, and a possibly singular nonlinearity f on $\Omega \times (0, \infty)$, we give sufficient conditions on f so that the p-Laplace equation $-\Delta_p u = f(x, u)$ admits a solution $u \in W_0^{1,p}(\Omega)$. On the basis of a comparison principle we will give a sufficient condition under which such a problem admits a unique solution.

Keywords: p-Laplacian; Singular boundary value problem; Comparison principle

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $f : \Omega \times (0, \infty) \to [0, \infty)$ be a given singular nonlinearity. In this paper we are concerned with existence and uniqueness of weak solutions of the following singular boundary value problem:

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian for $1 < p < \infty$. The feature that needs to be highlighted in the boundary value problem (1.1) is the possible singularity the nonlinearity $f(x, t)$ could exhibit when $t \to 0^+$. This type of problem has received a lot of attention and extensive investigations have been carried out over the years around the questions of existence, uniqueness, boundary behavior and regularity of solutions. Perhaps the earliest work related to singular boundary value problems is that of Fulks and Maybee [21]. However, it was the pioneering work of Crandall, Rabinowitz and Tartar [8] that has inspired an enormous amount of work in these and related problems. We refer the reader to the papers [6,7,9,10,13,17,20,22,30–32,34] for work related to singular boundary value problems when $p = 2$. In the paper [8], the authors study the existence of both classical and weak solutions of the above problem when the left-hand side is replaced by a linear second order elliptic operator that satisfies the maximum principle. In [23], Lazer and McKenna study the problem (1.1) for the $p = 2$ case, and when $f(x, t) = b(x)t^{-\gamma}$ with

E-mail address: amohammed@bsu.edu.

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$b \in C^\alpha(\overline{\Omega})$, $b > 0$ on $\overline{\Omega}$. They establish the existence of a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for any $\gamma > 0$, and go on to show that such a solution belongs to $W_0^{1,2}(\Omega)$ if and only if $\gamma < 3$. The problem is further studied in [20], where G. Gui and F.H. Lin investigate various levels of regularity of the solutions depending on the growth rates of $b(x)$ near the boundary $\partial\Omega$. In the paper [31], Shi and Yao consider the above problem with $p = 2$. Under general conditions, they prove existence and uniqueness of solutions to the problem (1.1). This paper is partly motivated by the paper of Lair and Shaker [22] where they study problem (1.1) with $p = 2$, and $f(x,t) = b(x)g(t)$. They show that if $b \in L^2(\Omega)$, $b \geqslant 0$, and g is a positive, nonincreasing function in $L^1(0,\delta)$ for some $\delta > 0$, then the problem (1.1) admits a unique solution in $W_0^{1,2}(\Omega)$. A particular case to which their result applies is $g(t) = t^{-\gamma}$ with $0 < \gamma < 1$. We will obtain this as a special case of a more general result which applies to more general class of functions f.

In the recent papers [27,28], Perera and Silva study the solvability of (1.1) where $f(x,t)$ satisfies various conditions. In [28] the authors allow very general f, but they consider solutions in $W_{\mathrm{loc}}^{1,p}(\Omega)$ that satisfy the boundary condition in (1.1) in a more general sense. We refer the reader to the paper for the exact conditions on f and the type of solutions they consider. We also point the reader to the papers [1,29] where similar problems are investigated. We should point out that the results contained in this paper are not contained in any of the above mentioned studies.

The present paper is organized as follows. In Section 2 we provide basic facts and recall some lemmas that will be needed later. We also specify the conditions on the nonlinearity f that will be used throughout the paper. Section 3 contains the main result on the existence of weak solutions of the problem (1.1). Several results that can be drawn from the main theorem are also given. The final section, Section 4, establishes a comparison principle from which uniqueness of solutions to the problem (1.1) follows.

2. Preliminaries

Throughout the paper the nonlinearity $f : \Omega \times (0,\infty) \to [0,\infty)$ will be assumed to be a Carathéodory function such that $f(\cdot,t)$ is in $C^\theta(\Omega)$ for some $0 < \theta < 1$. Unless specified otherwise we will also suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $C^{1,\varpi}$ boundary for some $0 < \varpi < 1$. The positive constants that will be used in estimations are not necessarily the same in every occurrence. In the sequel, further conditions will be stipulated on f.

Given $h \in W^{1,p}(\Omega)$, by a solution of the boundary value problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

we mean $u \in W^{1,p}(\Omega)$ such that $u - h \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = \int_\Omega f(x,u)\varphi \tag{2.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

A function $u \in W^{1,p}(\Omega)$ is called a sub-solution of (2.1) on Ω iff $(u-h)^+ \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \leqslant \int_\Omega f(x,u)\varphi \tag{2.3}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi \geqslant 0$ on Ω. A function $u \in W^{1,p}(\Omega)$ is called a super-solution of (2.1) on Ω iff $(u-h)^- \in W_0^{1,p}(\Omega)$ and (2.3) holds with the inequality reversed for all $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi \geqslant 0$ on Ω. Following usual practice, we shall write $u \leqslant h$ on $\partial\Omega$ to mean $(u-h)^+ \in W_0^{1,p}$.

The following comparison lemma, proved in [12,26], will also be useful.

Lemma 2.1. *Let $g(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in x and nondecreasing in t. Let $u, v \in W^{1,p}(\Omega)$ satisfy*

$$-\Delta_p u + g(x,u) \leqslant -\Delta_p v + g(x,v) \quad (x \in \Omega).$$

If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ on Ω.

Another lemma that we will need is the generalization of the Picone Identity to the p-Laplacian case. We refer the reader to the papers [2,16] for a proof.

Lemma 2.2 *(Picone's Identity). Let $v > 0$ and $u \geqslant 0$ be weakly differentiable. Denote*

$$L(u, v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla u \cdot \nabla v. \tag{2.4}$$

Then $L(u, v) \geqslant 0$, and $L(u, v) = 0$ a.e. on Ω if and only if $u = \alpha v$ for some constant α in each component of Ω.

In this paper we shall use the notation q for the Hölder conjugate of $p > 1$, that is $q := p/(p-1)$. Also we let p^* be the Sobolev conjugate exponent of p, namely $p^* = np/(n-p)$ if $1 < p < n$, and $p^* = \infty$ if $p \geqslant n$.

Before we state conditions on f that will be needed in the paper, let us identify a class $\mathcal{G}$ of functions $g : (0, \infty) \to (0, \infty)$ and a class $\mathcal{B}$ of non-negative functions b in $L^q(\Omega)$ such that the following conditions hold:

[G1] There is $\rho \geqslant 1$ and a positive constant C such that $g(t) \leqslant C$ for all $t \geqslant \rho$.
[G2] One of the conditions (1), (2), or (3) below holds, where $h(t) = tg(t)$, $t > 0$:
  (1) $h(t) \leqslant C$ for all $0 < t < \delta$ and some positive constants C and δ.
  (2) h is nondecreasing.
  (3) (i) g is nonincreasing.
    (ii) For each $\theta \in (0, 1)$ there is a constant $C_\theta \geqslant 1$ such that $g(\theta t) \leqslant C_\theta g(t)$ for all $t > 0$.
    (iii) There is $\omega \in C_0^1(\overline{\Omega})$ with $\omega > 0$ on Ω such that $bg(\omega) \in L^{p^*/(p^*-1)}(\Omega)$.

Remark 2.3. Condition 3(iii) of [G2] was used in the papers [1,27,29] with $g(t) = t^{-\gamma}$ for $\gamma > 0$.

We are now ready to list the conditions on f that will be used in this paper. In stating the conditions, we use the notation

$$\gamma_s(x) = \sup\{f(x, t)\colon\ t \geqslant s\} \quad (x \in \Omega),$$

for any $s > 0$.

[F1] $\gamma_s \in L^q(\Omega)$ for each $s > 0$.
[F2] There is $(g, b) \in \mathcal{G} \times \mathcal{B}$ that satisfies conditions [G1]–[G2], a measurable function a for which $\{x \in \Omega\colon\ 0 < a(x) \leqslant 1\}$ has a positive measure, and $0 \leqslant a \leqslant b$ such that
  (1) $f(x, s) \geqslant a(x)$ for $(x, s) \in \Omega \times (0, \rho^{-1})$,
  (2) $f(x, s) \leqslant b(x)g(s)$ for $(x, s) \in \Omega \times (\rho, \infty)$,
  (3) $f(x, s)g(t) \leqslant Cf(x, t)g(s)$ for $x \in \Omega$, $0 < s < t$ and some constant $C > 0$.

We start with the following lemma.

Lemma 2.4. *Let f satisfy* [F1] *and $h \in W^{1,p}(\Omega)$ with $h > 0$ on Ω. Let ψ_{sub} be a sub-solution and ψ_{sup} a super-solution of* (2.1) *on Ω. If $0 < \inf_\Omega \psi_{\text{sub}} \leqslant \psi_{\text{sub}} \leqslant \psi_{\text{sup}}$ on Ω then there is a solution $u \in W^{1,p}(\Omega)$ of* (1.1) *such that $\psi_{\text{sub}} \leqslant u \leqslant \psi_{\text{sup}}$ a.e. on Ω.*

Proof. The lemma is a consequence of [11, Theorem 4.14]. For completeness we include the short proof.
Take $0 < \ell = \inf_\Omega \psi_{\text{sub}}$ and let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x, t) := \begin{cases} f(x, \ell) & \text{if } t < \ell, \\ f(x, t) & \text{if } t \geqslant \ell. \end{cases}$$

Then $F(x, \cdot)$ is Hölder continuous on $\mathbb{R}$ for each $x \in \Omega$ and $|F(x, t)| \leqslant \gamma_\ell(x)$ on Ω. Consider now the Dirichlet problem

$$\begin{cases} -\Delta_p u = F(x, u) & (x \in \Omega), \\ u = h & (x \in \partial\Omega). \end{cases}$$

Note that ψ_{sub} is a sub-solution, and ψ_{sup} is a super-solution of this problem. Since $\gamma_\ell \in L^q(\Omega)$, it follows by [11, Theorem 4.14] that this problem admits a solution $u \in W^{1,p}(\Omega)$ such that $\psi_{\mathrm{sub}} \leqslant u \leqslant \psi_{\mathrm{sup}}$. Finally we note that u is a solution of (2.1) as claimed. □

3. Existence of weak solutions

We now show the existence of a solution of (1.1) in $W_0^{1,p}(\Omega)$.

Theorem 3.1. *Suppose that f satisfies* [F1]–[F2]. *Then problem* (1.1) *admits a solution in* $W_0^{1,p}(\Omega)$.

Proof. By replacing g and b by $C^{-1}g$ and Cb, respectively, if necessary, we can assume that $g(t) \leqslant 1$ for all $t \geqslant \rho$ in condition [G1]. For $k = 1, 2, \ldots,$ let ψ_k be the solution of

$$\begin{cases} -\Delta_p u(x) = a_k(x) & \text{in } \Omega, \\ u(x) = k^{-1} & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where, for $k = 1, 2, \ldots,$

$$a_k(x) := \min\left\{a(x), \frac{k+1}{k}\right\} \quad (x \in \Omega).$$

Likewise, let ψ_∞ be the solution of (3.1) with $k = \infty$. Note that since $a_k \in L^\infty(\Omega)$ the problem (3.1) does indeed have a solution, and that the solution belongs to $C^1(\overline{\Omega})$ (see [29, Proposition 2.1]). By the comparison lemma, we have $0 \leqslant \psi_\infty \leqslant \psi_k \leqslant \psi_1$ for all $k = 1, 2, \ldots$ and $\psi_k \geqslant k^{-1}$ on $\overline{\Omega}$ for all $k = 1, 2, \ldots.$ By the Strong Maximum Principle, [33], we note that $\psi_\infty > 0$ on Ω. Replacing ψ_k by $\epsilon\psi_k$ for an appropriate $0 < \epsilon \leqslant 1$, if necessary, we can assume that $\psi_k < \rho^{-1}$ on $\overline{\Omega}$ for all $k = 1, 2, 3, \ldots.$ Here ρ is the positive constant in conditions [G1] and [F2]. Note that after such possible modification we have $-\Delta_p\psi_k \leqslant a$ in Ω, and $\psi_k = \epsilon k^{-1}$ on $\partial\Omega$.

For each $k \in \mathbb{Z}^+$ let us now consider the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = f\big(x, u(x)\big) & (x \in \Omega), \\ u(x) = \epsilon k^{-1} & (x \in \partial\Omega). \end{cases} \tag{BVP_k}$$

By (1) of [F2] we observe that

$$-\Delta_p\psi_k - f(x, \psi_k) \leqslant -\Delta\psi_k - a(x) \leqslant 0$$

and thus ψ_k is a sub-solution of (BVP_k) for all k.

Now, let ψ be a solution of

$$\begin{cases} -\Delta_p u(x) = b(x) & \text{in } \Omega, \\ u(x) = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Note that (3.2) has a solution since $b \in L^q(\Omega)$, and that the solution is in $L^\infty(\Omega)$ if $q > p/n$. This follows from [19, Proposition 1.3] when $1 < p \leqslant n$ and from the Sobolev embedding theorem when $p > n$ (see [18, Theorem 7.10] for example). By comparison principle, we note that $\psi_k \leqslant \psi$ for all $k = 1, 2, \ldots$ and $\psi \geqslant 1$ on $\overline{\Omega}$. Put $\psi_0 = \rho\psi$, so that $\psi_0 \geqslant \rho$ on Ω. Then by (2) of [F2] we have

$$-\Delta_p\psi_0 - f(x, \psi_0) \geqslant -\Delta_p\psi_0 - b(x) \geqslant -\Delta_p\psi_0 - \rho^{p-1}b(x) = 0.$$

Thus ψ_0 is a super-solution of (BVP_k) on Ω for any $k = 1, 2, \ldots.$ Let us fix a positive integer m. By Lemma 2.4 let u_m be a solution of (BVP_m) such that $\psi_m \leqslant u_m \leqslant \psi_0$ on Ω. Note that u_m is a super-solution of (BVP_{m+1}) and therefore, again by Lemma 2.4, there is a solution u_{m+1} of (BVP_m) such that $\psi_{m+1} \leqslant u_{m+1} \leqslant u_m$. Continuing in this manner, we construct a sequence $\{u_k\}$ of solutions of (BVP_k) such that for all $k \geqslant m$

$$\psi_\infty \leqslant u_{k+1} \leqslant u_k \leqslant \cdots \leqslant u_m \leqslant \psi_0 \quad \text{in } \Omega.$$

Let us also note that $u_k \geqslant \epsilon k^{-1}$ on $\overline{\Omega}$. We define

$$u(x) = \lim_{k\to\infty} u_k(x) \quad (x \in \Omega). \tag{3.3}$$

We use $\varphi_k := u_k - \epsilon/k$ as a test function for the solution u_k on Ω (by condition [F1]). We obtain

$$\int_\Omega |\nabla u_k|^p = \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_k - \epsilon/k) = \int_\Omega f(x,u_k)(u_k - \epsilon/k) \leqslant \int_\Omega f(x,u_k)u_k. \tag{3.4}$$

We now use condition [F2] to show that there is a positive constant C, independent of k, such that

$$\int_\Omega f(x,u_k)u_k \leqslant C. \tag{3.5}$$

First we note that conditions (2) and (3) of [F2] imply that

$$f(x,u_k)u_k = \frac{f(x,u_k)}{g(u_k)}g(u_k)u_k \leqslant C\frac{f(x,\psi_0)}{g(\psi_0)}g(u_k)u_k.$$

If (1) of [G2] holds, then this together with [G1] implies

$$\frac{f(x,\psi_0)}{g(\psi_0)}g(u_k)u_k \leqslant Mb\max\{\rho,u_k\} \leqslant Mb\psi_0,$$

for some positive constant M.

If (2) of [G2] holds, then recalling $g(\psi_0) \leqslant 1$, it follows from (2) of [F2] that

$$\frac{f(x,\psi_0)}{g(\psi_0)}g(u_k)u_k \leqslant f(x,\psi_0)\psi_0 \leqslant b\psi_0.$$

Therefore, since $\psi_0 \in L^p(\Omega)$ and $b \in L^q(\Omega)$, the estimate (3.5) holds if either (1) or (2) of [G2] holds.

Suppose now that (3) of [G2] holds. By [29, Proposition 2.1] we note that $\psi_\infty > 0$ on Ω, and $\partial\psi_\infty/\partial\nu > 0$ on $\partial\Omega$, where ν is the unit inner normal. Thus, we have $\inf_\Omega(\psi_\infty/\omega) > 0$, where ω is as in condition 3(iii) of [G2]. That is $\theta\omega \leqslant \psi_\infty$ on Ω for some $0 < \theta \leqslant 1$. Then on using 3(i) and 3(ii) of [G2], we estimate

$$\frac{f(x,\psi_0)}{g(\psi_0)}g(u_k)u_k \leqslant \frac{f(x,\psi_0)}{g(\psi_0)}g(\theta\omega)\psi_0 \leqslant C_\theta b g(\omega)\psi_0.$$

Since $\psi_0 \in W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$, the assumption in 3(iii) of [G2] shows that estimate (3.5) holds in this case as well.

Thus estimates (3.4) and (3.5) show that the sequence $\{\varphi_k\}$ is bounded in $W_0^{1,p}(\Omega)$. We pick a subsequence, still denoted by $\{\varphi_k\}$, such that it converges weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and pointwise a.e. on Ω. Note that the sequence converges to $u \in W_0^{1,p}(\Omega)$, and that $\{u_k\}$ converges weakly to u in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. on Ω. Now let $\Omega_0 \Subset \Omega$. On Ω_0 we have

$$\big|\big(f(x,u_k) - f(x,u_j)\big)(u_k - u_j)\big| \leqslant 4\gamma_\ell(x)\psi_0,$$

where $\ell = \min_{\overline{\Omega}_0}\psi_\infty$, and hence proceeding as in [28] one can show that

$$\int_{\Omega_0} \big(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_j|^{p-2}\nabla u_j\big)\cdot\nabla(u_k - u_j) \to 0,$$

as $k, j \to \infty$. It then follows (see [29]) from this that

$$\int_{\Omega_0} |\nabla u_k - \nabla u_j|^p \to 0, \quad k,j \to \infty. \tag{3.6}$$

Since $\{u_k\}$ converges strongly to u in $L^p(\Omega_0)$, the limit (3.6) shows that the sequence $\{u_k\}$ is Cauchy in $W^{1,p}(\Omega_0)$, and hence converges to u in $W^{1,p}(\Omega_0)$. In conclusion, given any compact set $\Omega_0 \subset \Omega$, we can find a subsequence of $\{u_k\}$ that converges to u strongly in $W^{1,p}(\Omega_0)$. Let us take note of the following estimates. If $p \geqslant 2$, then by Hölder's inequality, we have

$$\int\limits_{\Omega_0} |\nabla u_k - \nabla u|\big(|\nabla u_k| + |\nabla u|\big)^{p-2} \leqslant \|u_k - u\|_{W^{1,p}(\Omega_0)} \bigg(\int\limits_{\Omega_0} \big(|\nabla u_k| + |\nabla u|\big)^{\frac{p(p-2)}{p-1}} \bigg)^{\frac{p-1}{p}}$$
$$\leqslant 2^{p-2} |\Omega_0|^{1/(p-1)} \|u_k - u\|_{W^{1,p}(\Omega_0)} \bigg(\int\limits_{\Omega_0} \big(|\nabla u_k|^p + |\nabla u|^p\big) \bigg)^{\frac{p-2}{p}}$$
$$\leqslant C\|u_k - u\|_{W^{1,p}(\Omega_0)}. \tag{3.7}$$

In the last inequality, we have used the boundedness of $\{u_k\}$ in $W^{1,p}(\Omega_0)$. Similarly,

$$\int\limits_{\Omega_0} |\nabla u_k - \nabla u|^{p-1} \leqslant |\Omega_0|^{\frac{1}{p}} \bigg(\int\limits_{\Omega_0} |\nabla u_k - \nabla u|^p \bigg)^{\frac{p-1}{p}} \leqslant C\|u_k - u\|^{p-1}_{W^{1,p}(\Omega_0)}. \tag{3.8}$$

We now recall some useful inequalities (see [15]) that hold for all $\xi, \zeta \in \mathbb{R}^n$:

$$\big||\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\big| \leqslant \begin{cases} C|\xi - \zeta|(|\xi| + |\zeta|)^{p-2} & \text{if } p \geqslant 2, \\ C|\xi - \zeta|^{p-1} & \text{if } 1 < p \leqslant 2, \end{cases} \tag{3.9}$$

where C is a positive constant independent of ξ and ζ. The estimates (3.7) and (3.8) together with the inequalities (3.9) show that

$$\lim_{k\to 0} \int\limits_{\Omega_0} \big||\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\big| = 0. \tag{3.10}$$

Now, let $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp}\varphi \subseteq \Omega_0 \Subset \Omega$. It then follows from the limit (3.10) that

$$\int\limits_{\Omega} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\varphi \to \int\limits_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi. \tag{3.11}$$

Since $|f(x, u_k)\varphi| \leqslant C\gamma_\ell(x)$ on Ω_0 and $\gamma_\ell \in L^1(\Omega)$, it follows that

$$\int\limits_{\Omega} f(x, u_k)\varphi \to \int\limits_{\Omega} f(x, u)\varphi. \tag{3.12}$$

Therefore (3.11) and (3.12) show that the identity (2.2) holds for all $\varphi \in C_0^\infty(\Omega)$. We proceed to show that it also holds for any $\varphi \in W_0^{1,p}(\Omega)$. So, suppose $w \in W_0^{1,p}(\Omega)$. We choose a sequence $\{\eta_k\}$ of non-negative functions in $C_0^\infty(\Omega)$ such that $\eta_k \to |w|$ in $W_0^{1,p}(\Omega)$. By going to a subsequence if necessary, we can assume that $\eta_k \to |w|$ a.e. on Ω. Then, by Fatou's lemma and Hölder's inequality we see that

$$\bigg|\int\limits_{\Omega} f(x, u)w\bigg| \leqslant \int\limits_{\Omega} f(x, u)|w| \leqslant \liminf_{k\to\infty} \int\limits_{\Omega} f(x, u)\eta_k$$
$$= \lim_{k\to\infty} \int\limits_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla\eta_k \leqslant \|u\|^{p-1}_{W_0^{1,p}(\Omega)} \lim_{k\to\infty} \|\eta_k\|_{W_0^{1,p}(\Omega)}$$
$$= \|u\|^{p-1}_{W_0^{1,p}(\Omega)} \|w\|_{W_0^{1,p}(\Omega)}.$$

Now, if $\varphi \in W_0^{1,p}(\Omega)$, and $\varphi_k \to \varphi$, then taking $w := \varphi_k - \varphi$ in the above inequality shows that

$$\lim_{k\to\infty} \int\limits_{\Omega} f(x, u)\varphi_k = \int\limits_{\Omega} f(x, u)\varphi.$$

We also have

$$\lim_{k\to\infty}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi_k=\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi.$$

It follows that (2.2) holds for all $\varphi\in W_0^{1,p}(\Omega)$. Thus $u\in W_0^{1,p}(\Omega)$ is a solution of (1.1) such that $\psi_\infty\leqslant u\leqslant\psi_0$ on Ω. □

As an immediate corollary we have the following, which reduces to Theorem 4 of [22] when $p=2$.

Corollary 3.2. *Let $g:(0,\infty)\to(0,\infty)$ be nonincreasing, and assume that $g\in L^1(0,\delta)$ for some $\delta>0$. If $f(x,t)=b(x)g(t)$ for some non-trivial and non-negative $b\in L^q(\Omega)$, then* (1.1) *has a solution.*

Proof. It is enough to realize that if g satisfies the hypothesis, then $tg(t)\leqslant C$ for all $0<t<\delta$ and some positive constant C. Thus, g satisfies both the conditions [G1] and (1) of [G2]. Therefore f satisfies both the conditions [F1]–[F2]. The corollary then follows from Theorem 3.1. □

Remark 3.3. Note that $g(t)=(t^\alpha\log^\beta(1+t))^{-1}$ where $0<\alpha<1$, and $\beta\geqslant 1-\alpha$ or when $\alpha=1$ and $\beta=0$ satisfies the conditions [G1] and (1) of [G2]. Therefore, if $f(x,t)=b(x)g(t)$ for some non-trivial and non-negative $b\in L^q(\Omega)$, then by Theorem 3.1, the problem (1.1) has a solution. However, since $g\notin L^1(0,\delta)$ for any $\delta>0$, Theorem 4 of [22] does not guarantee a solution in $W_0^{1,p}(\Omega)$ in the special case $p=2$. Similarly, $g(t)=1+(\sin^2(1/t))/t^\alpha$ satisfies [G1] and (1) of [G2] for any $0<\alpha\leqslant 1$, and therefore Theorem 3.1 applies in this case as well. Since g is not monotonic, this conclusion cannot be drawn from [22] when $p=2$.

For the next result, we will need the following condition on $g:(0,\infty)\to(0,\infty)$, where g is C^α, $0<\alpha<1$.

[G3] $\lim_{t\to 0^+}g(t)=\infty$.

Consider

$$G(t):=\int_t^d g(s)\,ds, \tag{3.13}$$

where $0<d\leqslant\infty$ is chosen such that the integral (3.13) is finite for $0<t<d$. We define $\psi:[0,d]\to[0,\psi(d)]$ to be the increasing function

$$\psi(t)=\int_0^t\frac{1}{G(s)^{1/p}}\,ds. \tag{3.14}$$

Let $c=\psi(d)$ and $\varphi:[0,c]\to[0,d]$ be the inverse of ψ. We note some properties of φ in the following remark.

Remark 3.4.

(1) φ satisfies

$$\begin{cases} -p|\varphi'|^{p-2}\varphi''=g(\varphi) & \text{in } (0,c),\\ \varphi(t)>0 & \text{in } (0,c],\\ \varphi(0)=0. \end{cases}$$

(2) From (1) we see that φ' is decreasing on $(0,c)$ and therefore $\varphi'(0+)\in(0,\infty]$. Furthermore φ is increasing on $(0,c]$.

(3) If, in addition g is nonincreasing, then on invoking Lemma 2.1 of [24] we see that

$$\lim_{t\to 0^+}\frac{G(t)}{g(t)}=0.$$

Therefore, in this case

$$\frac{-\varphi'(t)}{\varphi''(t)} = \frac{G(\varphi(t))}{g(\varphi(t))} \cdot \frac{1}{\varphi'(t)}$$

is bounded on $(0, c]$.

Let z be an eigenfunction of $-\Delta_p$ on Ω corresponding to the first eigenvalue λ_1 normalized so that $0 < z(x) < c$ for all $x \in \Omega$. It is known that $z \in C_0^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and that $|\nabla z| \neq 0$ on $\partial\Omega$. (See [4,25,26,33].)

Theorem 3.5. *Let $f(x,t) = b(x)g(t)$ where $b \in L^\infty(\Omega)$ with $\inf_\Omega b > 0$ and g satisfies* 3(i), 3(ii) *of* [G2] *and* [G3]. *Then problem* (1.1) *has a solution if and only if $\varphi(z)g(\varphi(z)) \in L^1(\Omega)$.*

Proof. Suppose $\varphi(z)g(\varphi(z)) \in L^1(\Omega)$. Since $\omega := \varphi(z) \in C_0^1(\overline{\Omega})$ it follows that conditions [G1] and 3(iii) of [G2] hold. Therefore the hypotheses of Theorem 3.1 are satisfied by f, and thus problem (1.1) has a solution. For the converse, let $w = \beta\varphi(z)$ where β is a positive number to be determined shortly. Then

$$|\nabla w|^{p-2}\nabla w = \beta^{p-1}\big(\varphi'(z)\big)^{p-1}|\nabla z|^{p-2}\nabla z.$$

Let $\eta \in C_0^\infty(\Omega)$. By Remark 3.4, and recalling that z is an eigenfunction of the p-Laplacian, direct computation shows that

$$\begin{aligned}\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\eta &= \beta^{p-1}\int_\Omega |\nabla z|^{p-2}\nabla z \cdot \nabla\big(\big(\varphi'(z)\big)^{p-1}\eta\big) - (p-1)\beta^{p-1}\int_\Omega |\nabla z|^p\big(\varphi'(z)\big)^{p-2}\varphi''(z)\eta \\ &= \beta^{p-1}\int_\Omega z^{p-1}\big(\varphi'(z)\big)^{p-1}\eta + \beta^{p-1}\frac{p-1}{p}\int_\Omega |\nabla z|^p g\big(\varphi(z)\big)\eta.\end{aligned}$$

Consequently we get

$$\begin{aligned}\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\eta - \int_\Omega bg(w)\eta &= \int_\Omega \eta g\big(\varphi(z)\big)\left[\beta^{p-1}z^{p-1}\frac{(\varphi'(z))^{p-1}}{g(\varphi(z))} + \beta^{p-1}\frac{p-1}{p}|\nabla z|^p - b\right] \\ &= \int_\Omega \eta g\big(\varphi(z)\big)\left[\beta^{p-1}\left(p^{-1}z^{p-1}\left(\frac{-\varphi'(z)}{\varphi''(z)}\right) + \frac{p-1}{p}|\nabla z|^p\right) - b\right].\end{aligned}$$

Note that, since $|\nabla z| \neq 0$ on $\partial\Omega$, we have

$$\inf_\Omega\left[z^{p-1}\left(\frac{-\varphi'(z)}{\varphi''(z)}\right) + \frac{p-1}{p}|\nabla z|^p\right] > 0,$$

and thus, by choosing $\beta \geqslant 1$ sufficiently big, we see that

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\eta \geqslant \int_\Omega bg\big(\varphi(z)\big)\eta \geqslant \int_\Omega bg(w)\eta$$

for all $\eta \in C_0^\infty(\Omega)$ with $\eta \geqslant 0$. A straight forward application of Fatou's lemma shows that the above inequality holds for all non-negative $\eta \in W_0^{1,p}(\Omega)$. Thus, if u is a solution of (1.1) by the comparison principle, Lemma 2.1, we have

$$u(x) \leqslant \beta\varphi\big(z(x)\big) \quad (x \in \Omega).$$

Using $\varphi := u$ in (2.2) we note that

$$\Big(\inf_\Omega b\Big)\int_\Omega g(u)u \leqslant \int_\Omega bg(u)u = \int_\Omega |\nabla u|^p < \infty.$$

Therefore since, by condition 3(ii) of [G2],

$$C_\beta\varphi(z)g\big(\varphi(z)\big) \leqslant \beta\varphi(z)g\big(\beta\varphi(z)\big) \leqslant ug(u) \quad \text{on } \Omega \tag{3.15}$$

for some positive constant C_β, we have

$$\int_\Omega \varphi(z)g(\varphi(z)) < \infty.$$

This proves the theorem. □

Corollary 3.6. *Let* $f(x,t) = b(x)t^{-\lambda}$ *for some* $b \in L^\infty(\Omega)$ *with* $\inf_\Omega b > 0$ *and* $\lambda \geqslant 0$. *Problem* (1.1) *has a solution iff* $\lambda < \frac{2p-1}{p-1}$.

Proof. If $0 \leqslant \lambda \leqslant 1$, then $g(t) = t^{-\lambda}$ satisfies conditions [G1] and (2) of [G2]. Therefore in this case, by the above theorem, problem (1.1) has a solution. So suppose $\lambda > 1$. Then we note conditions 3(i), 3(ii) of [G2] and [G3] hold for $g(t) = t^{-\lambda}$. Direct computation, and using $d = \infty$ in (3.13) shows that

$$\varphi(t) = Ct^{p/(\lambda+p-1)}$$

for some positive constant C that depends only on p and λ. Therefore we have

$$\varphi(z)g(\varphi(z)) = Cz^{-p(\lambda-1)/(\lambda+p-1)},$$

where z is a non-negative eigenfunction of $-\Delta_p$ for the Dirichlet problem that corresponds to the first eigenvalue λ_1. We recall that $z \in C^{1,\alpha}(\overline{\Omega})$, $z > 0$ on Ω, and that $\partial z/\partial\nu > 0$ on $\partial\Omega$, where ν is the inner normal vector field on $\partial\Omega$. But then, by a lemma of Lazer and McKenna in [23] (actually its proof to be exact) we conclude that

$$\int_\Omega z^{-p(\lambda-1)/(\lambda+p-1)}\,dx < \infty \quad \text{if and only if} \quad \lambda < \frac{2p-1}{p-1}.$$

Thus the corollary follows from Theorem 3.5 above. □

Remark 3.7.

(1) It was proved in [23] that $-\Delta u = b(x)u^{-\gamma}$ has a solution in $W_0^{1,2}(\Omega)$ if $b \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$ with $b(x) > 0$ on $\overline{\Omega}$ and $\gamma < 3$. This is a special case of Theorem 3.5 above with $p = 2$.
(2) A result related to Theorem 3.5 is proved in [34] for the case $p = 2$ under a different set of assumptions on b and g.

4. A comparison principle and uniqueness

In this section, we employ a combination of techniques developed in [3] and [14] to prove a comparison principle for solutions of (1.1) under less stringent conditions than previously considered. The paper [3] uses a generalization of a differential inequality due to Picone to prove Sturm type comparison results. Inspired by the paper [5], J. Diaz and J. Saa establish existence and uniqueness of solutions to (1.1) where the right-hand side is non-singular. More specifically, they consider $f : \Omega \times [0,\infty) \to \mathbb{R}$ such that $t \mapsto f(x,t)$ is continuous on $[0,\infty)$ for a.e $x \in \Omega$, and $x \mapsto f(x,t)$ is in $L^\infty(\Omega)$ for each $t \geqslant 0$. Furthermore, f is required to satisfy condition [F3] below.

We will assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, and we do not assume any regularity of the boundary. Let us now consider a measurable nonlinearity $f : \Omega \times (0,\infty) \to [0,\infty)$ that satisfies the following condition.

[F3] For each $x \in \Omega$, the function $t \mapsto f(x,t)t^{1-p}$ is decreasing on $(0,\infty)$.

In the next theorem, we take two measurable functions $\psi_j : \Omega \times (0,\infty) \to [0,\infty)$ such that

$$\psi_1(x,z) \leqslant \psi_2(x,z) \quad \text{for all } (x,z) \in \Omega \times (0,\infty). \tag{4.1}$$

Theorem 4.1 *(A Comparison Principle). Suppose ψ_1, ψ_2 satisfy* (4.1) *and let ψ_1 (or ψ_2) satisfy* [F3]. *Furthermore let $u, v \in W^{1,p}(\Omega)$, with $u \in L^\infty(\Omega)$, $u > 0$, $v > 0$ on Ω be such that*

$$-\Delta_p u \leqslant \psi_1(x,u) \quad \text{and} \quad -\Delta_p v \geqslant \psi_2(x,v) \quad \text{on } \Omega. \tag{4.2}$$

If $u \leqslant v$ on $\partial\Omega$ and $\psi_1(x,u)$ (or $\psi_2(x,u)$) belongs to $L^1(\Omega)$, then $u \leqslant v$ on Ω.

Proof. Suppose to the contrary that $\Omega_0 := \{x \in \Omega\colon u(x) > v(x)\}$ has positive measure. For $0 < \epsilon < 1$, let $u_\epsilon = u + \epsilon$, and $v_\epsilon = v + \epsilon$, and for each $k \geqslant 1 + \|u\|_{L^\infty(\Omega)}$, let

$$w_{\epsilon,k} = u_\epsilon^p - (v_\epsilon \wedge k)^p.$$

We note that $\epsilon \leqslant u_\epsilon \leqslant k$ and $\epsilon \leqslant v_\epsilon \wedge k \leqslant k$ on Ω. Moreover,

$$0 \leqslant w_{\epsilon,k}^+ \leqslant M(u-v)^+$$

for some positive constant M that depends on p and k. Since $u \leqslant v$ on $\partial\Omega$, it follows that $w_{\epsilon,k}^+ \in W_0^{1,p}(\Omega)$ and therefore both $u_\epsilon^{1-p} w_{\epsilon,k}^+$ and $(v_\epsilon \wedge k)^{1-p} w_{\epsilon,k}^+$ belong to $W_0^{1,p}(\Omega)$. Here $w_{\epsilon,k}^+ = \max\{0, w_{\epsilon,k}\}$.

From (4.2) we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{w_{\epsilon,k}^+}{u_\epsilon^{p-1}}\right) \leqslant \int_\Omega \psi_1(x,u)\frac{w_{\epsilon,k}^+}{u_\epsilon^{p-1}}, \tag{4.3}$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{w_{\epsilon,k}^+}{(v_\epsilon \wedge k)^{p-1}}\right) \geqslant \int_\Omega \psi_2(x,v)\frac{w_{\epsilon,k}^+}{(v_\epsilon \wedge k)^{p-1}}. \tag{4.4}$$

We compute and see that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{w_{\epsilon,k}^+}{u_\epsilon^{p-1}}\right) = \int_{\Omega_0} |\nabla u|^p - p\left(\frac{v_\epsilon}{u_\epsilon}\right)^{p-1}|\nabla u|^{p-2}\nabla u \cdot \nabla v + (p-1)\int_{\Omega_0}\left(\frac{v_\epsilon}{u_\epsilon}\right)^p |\nabla u|^p,$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{w_{\epsilon,k}^+}{(v_\epsilon \wedge k)^{p-1}}\right) = \int_{\Omega_0} -|\nabla v|^p + p\left(\frac{u_\epsilon}{v_\epsilon}\right)^{p-1}|\nabla v|^{p-2}\nabla v \cdot \nabla u - (p-1)\int_{\Omega_0}\left(\frac{u_\epsilon}{v_\epsilon}\right)^p |\nabla v|^p.$$

Let us now assume that ψ_1 satisfies (4.1) and that $\psi_1(x,u) \in L^1(\Omega)$. The proof is the same if these assumptions are made for ψ_2 instead of ψ_1.

We now subtract (4.4) from (4.3), and we use the last two equations above to obtain the following estimates:

$$\begin{aligned}\int_{\Omega_0} L(u_\epsilon, v_\epsilon) + L(v_\epsilon, u_\epsilon) &\leqslant \int_{\Omega_0}\left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}} - \frac{\psi_2(x,v)}{(v_\epsilon \wedge k)^{p-1}}\right) w_{\epsilon,k} \leqslant \int_{\Omega_0}\left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}} - \frac{\psi_1(x,v)}{(v_\epsilon \wedge k)^{p-1}}\right) w_{\epsilon,k} \\ &= \int_\Omega \left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}} - \frac{\psi_1(x,v)}{(v_\epsilon \wedge k)^{p-1}}\right) w_{\epsilon,k}^+.\end{aligned}$$

Note that we have used (4.1) in obtaining the second inequality. Since $L(u_\epsilon, v_\epsilon) + L(v_\epsilon, u_\epsilon) \geqslant 0$ a.e. on Ω by Picone's identity, Lemma 2.4, we conclude that

$$\int_\Omega \left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}} - \frac{\psi_1(x,v)}{(v_\epsilon \wedge k)^{p-1}}\right) w_{\epsilon,k}^+ \geqslant 0.$$

Note that

$$\left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}} - \frac{\psi_1(x,v)}{(v_\epsilon \wedge k)^{p-1}}\right) w_{\epsilon,k}^+ \leqslant u\psi_1(x,u).$$

Since $u\psi_1(x,u) \in L^1(\Omega)$, we use Fatou's lemma to get

$$\limsup_{\epsilon\to 0}\int_\Omega\left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}}-\frac{\psi_1(x,v)}{(v_\epsilon\wedge k)^{p-1}}\right)w_{\epsilon,k}^+\leqslant\int_\Omega\limsup_{\epsilon\to 0}\left[\left(\frac{\psi_1(x,u)}{u_\epsilon^{p-1}}-\frac{\psi_1(x,v)}{(v_\epsilon\wedge k)^{p-1}}\right)w_{\epsilon,k}^+\right]$$
$$=\int_\Omega\left(\frac{\psi_1(x,u)}{u^{p-1}}-\frac{\psi_1(x,v)}{(v\wedge k)^{p-1}}\right)\left(u^p-(v\wedge k)^p\right)^+.$$

Therefore for all $k\geqslant 1+\|u\|_{L^\infty(\Omega)}$ we have

$$\int_\Omega\left(\frac{\psi_1(x,u)}{u^{p-1}}-\frac{\psi_1(x,v)}{(v\wedge k)^{p-1}}\right)\left(u^p-(v\wedge k)^p\right)^+\geqslant 0.$$

Another application of Fatou's lemma shows

$$\int_\Omega\left(\frac{\psi_1(x,u)}{u^{p-1}}-\frac{\psi_1(x,v)}{v^{p-1}}\right)\left(u^p-v^p\right)^+\geqslant 0. \tag{4.5}$$

But, since ψ_1 satisfies [F3], we note that

$$\left(\frac{\psi_1(x,u)}{u^{p-1}}-\frac{\psi_1(x,v)}{v^{p-1}}\right)\left(u^p-v^p\right)^+$$

is non-positive on Ω and is negative on $\Omega_0=\{x\in\Omega\colon u(x)>v(x)\}$. Thus

$$\int_\Omega\left(\frac{\psi_1(x,u)}{u^{p-1}}-\frac{\psi_1(x,v)}{v^{p-1}}\right)\left(u^p-v^p\right)^+<0$$

and this is an obvious contradiction to (4.5). □

Remark 4.2. A result similar to Theorem 4.1 is derived in [17,31] for $C^2(\Omega)\cap C(\overline{\Omega})$ solutions of (1.1) with $p=2$.

Corollary 4.3. *Suppose f satisfies* [F1], [F3]. *If u and v are bounded solutions of the Dirichlet problem* (1.1) *on Ω, such that $f(x,u)$ and $f(x,v)$ belong to $L^1(\Omega)$, then $u=v$ on Ω.*

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